

# The role of domination and smoothing conditions in the theory of eventually positive semigroups

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## Abstract

We perform an in-depth study of some domination and smoothing properties of linear operators and of their role within the theory of eventually positive operator semigroups. On the one hand we prove that, on many important function spaces, they imply compactness properties. On the other hand, we show that these conditions can be omitted in a number of Perron–Frobenius type spectral theorems. We furthermore prove a Kreĭn–Rutman type theorem on the existence of positive eigenvectors and eigenfunctionals under certain eventual positivity conditions.

## 1 Introduction

The solution of a linear autonomous evolution equation is often described by means of a  $C_0$ -semigroup on a Banach space, usually some kind of functions space. While, in many models, one expects the solution semigroup to be *positive*, that is, solutions with positive initial conditions remain positive, there are also examples which exhibit a more subtle type of positive behaviour. For example, it was noted in [9] and [10] that the solution semigroup of the bi-harmonic heat equation on  $\mathbb{R}^d$ , while not being positive, behaves in some sense *eventually* positive. This observation complemented earlier results on the corresponding elliptic problem; see for instance [14, 15, 16] and the references therein and also the recent paper

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[22]. A similar phenomenon occurs for the semigroup generated by the Dirichlet-to-Neumann operator on a two-dimensional disk as shown in [2].

These observations suggest that a general theory of eventually positive  $C_0$ -semigroups would be useful. While, in finite dimensions, such a theory has been developed during the last decade (see for instance [19, 20], [6, Theorem 2.9] and [8]), a systematic study of this phenomenon in infinite dimensions was initiated only recently in [4, 3]. Several spectral results for infinite dimensional operators with eventually positive powers were recently proved by the second author in [12], after eventually positive matrix powers had been intensively studied for at least two decades; see the introduction of [12] for references and additional details.

**A domination and a smoothing condition** In the present note we are mainly concerned with two conditions appearing in various characterisation theorems in [3]. The conditions involve the *principal ideal*  $E_u$  generated by some element  $u$  of the positive cone  $E_+$  of a real or complex Banach lattice  $E$ . That principal ideal is defined by

$$E_u := \{f \in E : \exists c \geq 0 \ |f| \leq cu\}.$$

It is a subspace of  $E$  and when equipped with the *gauge norm*  $\|\cdot\|_u$  given by

$$\|f\|_u := \inf\{c \geq 0 : |f| \leq cu\} \quad (1.1)$$

a Banach lattice in its own right. We will often assume that  $u \in E_+$  is a *quasi-interior point* of the positive cone, that is, a point such that  $E_u$  is dense in  $E$ . We refer to [18, 21] for the general theory of Banach lattices.

First condition: Given a linear operator  $A: E \supseteq D(A) \rightarrow E$  we refer to

$$D(A) \subseteq E_u \quad (\text{Dom})$$

as the *domination condition*. It plays an important role in the characterisation of eventually positive behaviour of the resolvent of  $A$  in [3, Theorem 4.4]. We call this a domination condition since for every  $v \in D(A)$  it implies the existence of  $c > 0$  such that  $|v| \leq cu$ .

Second condition: If  $A$  generates a  $C_0$ -semigroup  $(e^{tA})_{t \geq 0}$  on  $E$ , then we refer to

$$\exists t_0 \geq 0 : \quad e^{t_0 A} E \subseteq E_u \quad (\text{Smo})$$

as the *smoothing condition*. This condition is an important assumption in [3, Theorem 5.2] which characterises eventual positivity of  $(e^{tA})_{t \geq 0}$  by means of Perron–Frobenius like properties. We call (Smo) a smoothing condition since in general the gauge norm is stronger than the norm induced by on  $E$ , and also because  $(E_u, \|\cdot\|_u)$  is isometrically Banach lattice isomorphic to the space of real- or complex-valued continuous functions on some compact Hausdorff space  $K$ . The latter follows from the corollary to [21, Proposition II.7.2] and from Kakutani’s representation theorem for AM-spaces [18, Theorem 2.1.3].

If  $E$  is the space of real- or complex-valued continuous functions on a compact Hausdorff space  $K$ , endowed with the supremum norm, then we always have  $E_u = E$ . Hence, conditions (Dom) and (Smo) are automatically fulfilled on

such spaces. On many other Banach lattices, however, both conditions are quite strong. In a typical application we can think of  $E$  as an  $L^p$ -space over a bounded domain  $\Omega \subseteq \mathbb{R}^d$  with  $1 < p < \infty$  and of  $A$  as a differential operator, defined on an appropriate Sobolev space. The vector  $u$  could, for instance, be the constant function with value 1 in which case  $E_u$  coincides with  $L^\infty(\Omega)$ . In this case the domination condition **(Dom)** means that all functions in the domain of  $A$  are bounded; it is fulfilled if an appropriate Sobolev embedding theorem holds. The smoothing condition **(Smo)** means that the semigroup operator  $e^{t_0 A}$  maps every function to a bounded function, that is, it “smooths” unbounded initial data in some sense, see also the comment above.

It should be noted that, for analytic semigroups, condition **(Dom)** implies **(Smo)**; see [13, Remark 9.3.4] or the proof of [3, Corollary 5.3] for details and for a slightly stronger assertion. Given the fact that the assumptions **(Dom)** and **(Smo)** are fulfilled in many applications, they were not studied in much detail in [3]; it was merely demonstrated in [3, Example 5.4] that these conditions cannot be dropped in [3, Theorems 4.4 and 5.2] without one of the implications in those theorems failing.

**Aim of this note** The paper is devoted to an in-depth study of the conditions **(Dom)** and **(Smo)**. While, on spaces of continuous functions over a compact space, both conditions are always fulfilled, we will show in Section 2 that both conditions are rather strong on other function spaces such as the  $L^p$ -spaces. When  $p \in [1, \infty)$ , we see in Corollary 2.5 that condition **(Smo)** forces the semigroup  $(e^{tA})_{t \geq 0}$  to be eventually compact.

In Section 3 we present a short intermezzo on the existence of positive eigenvectors complementing earlier results in [4, Theorem 7.7.(i)]. In Sections 4 and 5 we show that some of the implications in the characterisation results in [3, Theorems 4.4 and 5.2] remain true without the conditions **(Dom)** and **(Smo)**.

**Eventual positivity: terminology** Several notions of eventual positivity were discussed in [4] and [3], some of which we recall for the convenience of the reader. For a concise formulation we introduce some notation. Let  $E$  be a real or complex Banach lattice. As usual we call  $f \in E$  *positive* if  $f \geq 0$ , and we write  $f > 0$  if  $f \geq 0$  but  $f \neq 0$ . If  $u, f \in E_+$ , then we write  $f \gg_u 0$  if there exists  $\varepsilon > 0$  such that  $f \geq \varepsilon u$ ; in this case we call  $f$  *strongly positive with respect to  $u$* . By  $\mathcal{L}(E)$  we denote the space of bounded linear operators on  $E$ . An operator  $T \in \mathcal{L}(E)$  is called *positive*, which we denote by  $T \geq 0$ , if  $TE_+ \subseteq E_+$ . We call  $T$  *strongly positive* with respect to a vector  $u \in E_+$  if  $Tf \gg_u 0$  for every  $0 < f \in E_+$ .

Now, let  $E$  be a complex Banach lattice with real part  $E_{\mathbb{R}}$  and let  $A: D(A) \rightarrow E$  be a linear operator. The operator  $A$  is called *real* if  $D(A) = E_{\mathbb{R}} \cap D(A) + iE_{\mathbb{R}} \cap D(A)$  and if  $A$  maps  $E_{\mathbb{R}} \cap D(A)$  to  $E_{\mathbb{R}}$ . The first notion of eventual positivity which we recall relates to the resolvent of  $A$ . We recall that the resolvent  $\lambda \mapsto \mathcal{R}(\lambda, A) := (\lambda I - A)^{-1} \in \mathcal{L}(E)$  is an analytic map on the *resolvent set*  $\rho(A)$ . We denote the spectrum of  $A$  by  $\sigma(A)$ .

**Definition.** Let  $A: E \supseteq D(A) \rightarrow E$  be a linear operator on a complex Banach

lattice  $E$  and let  $u \in E$  be a quasi-interior point of  $E_+$ . Let  $\lambda_0 \in \mathbb{R} \cap \sigma(A)$  be an isolated spectral value of  $A$ .

- (a) The resolvent  $\mathcal{R}(\cdot, A)$  is called *individually eventually strongly positive with respect to  $u$  at  $\lambda_0$*  if, for every  $0 < f \in E$ , there exists a  $\lambda_1 > \lambda_0$  with the following properties:  $(\lambda_0, \lambda_1] \subseteq \rho(A)$  and  $\mathcal{R}(\lambda, A)f \gg_u 0$  for all  $\lambda \in (\lambda_0, \lambda_1]$ .
- (b) The resolvent  $\mathcal{R}(\cdot, A)$  is called *individually eventually strongly negative with respect to  $u$  at  $\lambda_0$*  if, for every  $0 < f \in E$ , there exists a  $\lambda_1 < \lambda_0$  with the following properties:  $[\lambda_1, \lambda_0) \subseteq \rho(A)$  and  $-\mathcal{R}(\lambda, A)f \gg_u 0$  for all  $\lambda \in [\lambda_1, \lambda_0)$ .

We speak of *individual* eventual positivity as  $\lambda_1$  can depend on  $f$ . One can, of course, also define *uniform* eventual positivity; see [3, Definitions 4.1 and 4.2] for details. Note that if  $\mathcal{R}(\cdot, A)$  is eventually positive or negative at some  $\lambda_0 \in \sigma(A)$ , then  $A$  is *real*, that is,  $A$  leaves the real part  $E_{\mathbb{R}}$  of  $E$  invariant.

The above definitions make sense even if  $\lambda_0$  is not necessarily an isolated point of  $\sigma(A)$ , see [3, Definitions 4.1 and 4.2], but the above definition is sufficient for our purposes. In fact we will usually assume that  $\lambda_0$  is a pole of the resolvent  $\mathcal{R}(\cdot, A)$  as an analytic map on  $\rho(A)$ . Such a pole is always an eigenvalue of  $A$  as seen in [24, Theorem 2 in Section VIII.8], and the pole is of order one if and only if the geometric and algebraic multiplicities of  $\lambda_0$  as an eigenvalue of  $A$  coincide.

We next deal with  $C_0$ -semigroup on  $E$  generated by an operator  $A$  and denoted by  $(e^{tA})_{t \geq 0}$ .

**Definition.** Let  $(e^{tA})_{t \geq 0}$  be a  $C_0$ -semigroup on a complex Banach lattice  $E$  and let  $u \in E$  be a quasi-interior point of  $E_+$ . The semigroup  $(e^{tA})_{t \geq 0}$  is called *individually eventually strongly positive with respect to  $u$*  if, for every  $0 < f \in E$ , there exists a time  $t_0 \geq 0$  such that  $e^{tA}f \gg_u 0$  for all  $t \geq t_0$ .

We talk about *uniform eventual positivity* if  $t_0$  can be chosen independently of  $f \in E_+$ , see [3, Definition 5.1] for details. It is not difficult to see that  $A$  is a real operator if and only if the operator  $e^{tA}$  is real for every  $t \in [0, \infty)$ .

To a great extent the long-term behaviour of the semigroup is determined by properties relating to the *spectral bound*  $s(A) := \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\} \in [-\infty, \infty]$  of  $A$ . If  $s(A) \in (-\infty, \infty)$ , then of particular importance is the *peripheral spectrum* of  $A$  given by  $\sigma_{\text{per}}(A) := \{\lambda \in \sigma(A) : \operatorname{Re} \lambda = s(A)\}$  and the existence of a *dominant spectral value*, that is,  $\lambda_0 \in \sigma(A)$  such that  $\sigma_{\text{per}}(A) = \{\lambda_0\}$ .

In Section 3 we will also encounter a slightly weaker notion of eventual positivity; see Corollaries 3.2 and 3.3 and the preceding discussions.

We complete this section by clarifying some notation we will use throughout. The *dual space* of a real or complex Banach lattice  $E$  is denoted by  $E'$ ; it is also a Banach lattice and its positive cone  $E'_+$  is called the *dual cone* of  $E_+$ . A vector  $\varphi \in E'$  is positive if and only if  $\langle \varphi, f \rangle \geq 0$  for all  $0 \leq f \in E$ . Since  $E'$  is a Banach lattice, all the notation introduced above implies to the elements of this space, too; in particular, we write  $\varphi > 0$  if a functional  $\varphi \in E'$  fulfils  $\varphi \geq 0$  but  $\varphi \neq 0$ . We call the functional  $\varphi \in E'$  *strictly positive* if  $\langle \varphi, f \rangle > 0$  for all  $0 < f \in E$ . Note that every quasi-interior point of  $E'_+$  is a strictly positive functional, but the converse is not in general true. If  $A : E \supseteq D(A) \rightarrow E$  is a densely defined linear operator, then its dual operator is denoted by  $A' : E' \supseteq D(A') \rightarrow E'$ .

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## 2 Domination, smoothing and compactness

In this section we show that, on certain types of Banach lattices, the conditions (Dom) and (Smo) have rather strong consequences. Let  $E$  be a complex Banach lattice, let  $u \in E_+$ . The fact that the gauge norm on  $E_u$  is stronger than the induced norm from  $E$  has severe consequences on every operator  $T \in \mathcal{L}(E)$  which maps  $E$  to  $E_u$  as we shall see in the main theorems of this section.

To state the theorems we need to recall that a complex Banach lattice  $E$  is said to have *order continuous norm* if its real part  $E_{\mathbb{R}}$  has order continuous norm. We refer to [18, Definition 2.4.1] for a precise definition. We recall that every  $L^p$ -space with  $1 \leq p < \infty$  has order continuous norm, as has the space  $c_0$  of all real- or complex-valued sequences which converge to 0 endowed with the supremum norm. The space of continuous functions on a compact Hausdorff space  $K$  has never order continuous norm unless  $K$  is finite. We start with a lemma.

**Lemma 2.1.** *Let  $E$  be a real or complex Banach lattice with order continuous norm and let  $u \in E_+$ .*

- (i) *If  $T \in \mathcal{L}(E, E_u)$ , then  $T \in \mathcal{L}(E)$  is weakly compact.*
- (ii) *If  $T \in \mathcal{L}(E, E_u)$  is weakly compact, then  $T \in \mathcal{L}(E)$  is compact.*

*Proof.* It suffices to prove the lemma in case that the scalar field is real. Since  $E$  has order continuous norm every order interval in  $E$  is weakly compact; see [18, Theorem 2.4.2]. By definition of the gauge norm (1.1) every bounded set in  $E_u$  is contained in an order interval in  $E$ . Hence the natural injection  $j: E_u \rightarrow E$  given by  $j(x) = x$  is weakly compact. If we are precise, then  $T \in \mathcal{L}(E)$  is the composition  $j \circ T$ .

(i) As  $T: E \rightarrow E_u$  is bounded and  $j: E_u \rightarrow E$  is weakly compact we conclude that  $T: E \rightarrow E$  is weakly compact.

(ii) Because  $E_u$  is a Dunford-Pettis space and  $j \in \mathcal{L}(E_u, E)$  is weakly compact,  $j$  is a Dunford-Pettis operator, that is,  $x_n \rightarrow 0$  weakly in  $E_u$  implies that  $x_n \rightarrow 0$  in  $E$ ; see Definition 3.7.6, Proposition 3.7.9 and Proposition 1.2.13 in [18]. Let now  $(x_n)$  be a bounded sequence in  $E$ . Then by the weak compactness of  $T$  and the Eberlein-Šmulian theorem [5, Theorem V.6.1], we can find a subsequence  $(x_{n_k})$  such that  $Tx_{n_k} \rightarrow y$  weakly in  $E_u$  for some  $y \in E_u$ . Using that  $j \in \mathcal{L}(E_u, E)$  is a Dunford-Pettis operator we conclude that  $(j \circ T)x_{n_k} \rightarrow Ty$  in  $E$ . This proves that  $T: E \rightarrow E$  is a compact operator.  $\square$

**Theorem 2.2.** *Let  $E$  be a real or complex Banach lattice with order continuous norm and let  $u \in E_+$ . If  $T_k \in \mathcal{L}(E)$  and  $T_k E \subseteq E_u$  for  $k \in \{1, 2\}$ , then  $T_2 T_1 \in \mathcal{L}(E)$  is compact.*

*Proof.* First note that due to the closed graph theorem,  $T_k \in \mathcal{L}(E, E_u)$  for  $k = 1, 2$ , where  $E_u$  is as usual endowed with the gauge norm. By Lemma 2.1(i)  $T_1 \in \mathcal{L}(E)$  is weakly compact. As  $T_2: E \rightarrow E_u$  is continuous it is also weakly continuous and hence the composition  $T_2T_1: E \rightarrow E_u$  is weakly compact. Now Lemma 2.1(ii) implies that  $T_2T_1 \in \mathcal{L}(E)$  is compact.  $\square$

As a special case we can consider one operator  $T_1 = T_2 = T$ . If we assume that  $E$  is reflexive, then we obtain an even stronger result. Examples for reflexive Banach lattices are the  $L^p$ -spaces with  $1 < p < \infty$  on an arbitrary measure space.

**Theorem 2.3.** *Let  $E$  be a real or complex Banach lattice and let  $u \in E_+$ . Suppose that  $T \in \mathcal{L}(E)$  and that  $TE \subseteq E_u$ . Then the following assertions are true.*

- (i) *If  $E$  has order continuous norm, then  $T^2 \in \mathcal{L}(E)$  is compact.*
- (ii) *If  $E$  is reflexive, then  $T \in \mathcal{L}(E)$  is compact.*

*Proof.* (i) This is an obvious consequence of Theorem 2.2 taking  $T_1 = T_2 = T$ .

(ii) First note that due to the closed graph theorem,  $T \in \mathcal{L}(E, E_u)$ . If  $E$  is reflexive, then by the Banach-Alaoglu theorem every bounded set in  $E$  is contained in a weakly compact set. As  $T \in \mathcal{L}(E, E_u)$  is continuous and thus weakly continuous it follows that  $T \in \mathcal{L}(E, E_u)$  is weakly compact. Since it follows from [18, Theorem 2.4.2(v)] that every reflexive Banach lattice has order continuous norm, we can now apply Lemma 2.1(ii) which shows that  $T \in \mathcal{L}(E)$  is compact.  $\square$

In [3, Theorems 4.4 and 5.2] it was always assumed that certain spectral values of  $A$  be poles of the resolvent. In the corollaries below we will show that the above results imply that such assumptions are automatically satisfied if  $E$  has order continuous norm and if one of the conditions (Dom) or (Smo) is fulfilled. It is worthwhile pointing out that the assumption of the first corollary is a bit more general than condition (Dom).

**Corollary 2.4.** *Let  $E$  be a complex Banach lattice,  $u \in E_+$  and let  $A: E \supseteq D(A) \rightarrow E$  be a linear operator with non-empty resolvent set. Suppose that  $D(A^n) \subseteq E_u$  for some  $n \in \mathbb{N}$ . Then the following assertions are true.*

- (i) *If  $E$  has order continuous norm, then  $\mathcal{R}(\lambda, A)^{2n}$  is compact for every  $\lambda \in \rho(A)$ .*
- (ii) *If  $E$  is reflexive, then  $\mathcal{R}(\lambda, A)^n$  is compact for every  $\lambda \in \rho(A)$ .*

*In either case, all spectral values of  $A$  are poles of the resolvent  $\mathcal{R}(\cdot, A)$  and have finite algebraic multiplicity.*

*Proof.* Let  $\lambda \in \rho(A)$ . Then  $\mathcal{R}(\lambda, A)^n E = D(A^n) \subseteq E_u$ . Now Theorem 2.3(i) and (ii) yield (i) and (ii) respectively. In either case [23, Theorem 5.8-F] implies that all spectral values of  $A$  are poles of  $\mathcal{R}(\cdot, A)$  and have finite algebraic multiplicity.  $\square$



Corollary 2.4 is useful to prove that an operator in a concrete application has an eventually positive resolvent. This can often be done by using [3, Theorem 4.4]. As it turns out, if  $E$  has order continuous norm and the domination condition (Dom) is fulfilled, then, as a consequence of Corollary 2.4, some of the spectral theoretic assumptions in [3, Theorem 4.4] are automatically satisfied.

**Corollary 2.5.** *Let  $E$  be a complex Banach lattice with order continuous norm,  $u \in E_+$  and let  $(e^{tA})_{t \geq 0}$  be a  $C_0$ -semigroup on  $E$ . Suppose that  $e^{t_0 A} E \subseteq E_u$  for some  $t_0 \geq 0$ .*

*Then the semigroup  $(e^{tA})_{t \geq 0}$  is eventually compact. In particular, all spectral values of  $A$  are poles of the resolvent  $\mathcal{R}(\cdot, A)$  and have finite algebraic multiplicity. Moreover, the peripheral spectrum of  $A$  is finite.*

*Proof.* The semigroup is eventually compact since Theorem 2.3 implies that the operator  $e^{2t_0 A}$  is compact. Hence, according to [7, Corollary V.3.2], all spectral values of  $A$  are poles of  $\mathcal{R}(\cdot, A)$  and have finite algebraic multiplicity. It now follows from [7, Theorem II.4.18] that the peripheral spectrum of  $A$  is finite.  $\square$

As similar comment as given after Corollary 2.4 also applies here. In order to show that a given semigroup is eventually positive, one can combine results from [3, Section 5] with Corollary 2.5.

### 3 The existence of positive eigenvectors

This section is devoted to a Kreĭn–Rutman type theorem about the existence of positive eigenvectors. For eventually positive semigroup, a related result was given in [4, Theorem 7.7(i)]. Similar results for eventually and asymptotically positive operators can be found in [12, Section 6]. The latter results also contain existence results about positive eigenvectors of the dual operator. The following theorem and its corollaries are in the spirit of this latter result. The proof of Theorem 3.1 is inspired by the proofs of [4, Theorem 7.7(i)] and [12, Theorem 6.1].

**Theorem 3.1.** *Let  $A: E \supseteq D(A) \rightarrow E$  be a linear operator on a complex Banach lattice  $E$  and let  $\lambda_0 \in \sigma(A) \cap \mathbb{R}$  be a pole of the resolvent  $\mathcal{R}(\cdot, A)$ . Suppose that we have, for every  $f \in E_+$ ,*

$$(\lambda - \lambda_0) \text{dist}(\mathcal{R}(\lambda, A)f, E_+) \rightarrow 0 \quad (3.1)$$

*as  $\lambda \downarrow \lambda_0$ . Then the following assertions hold:*

- (i) *The number  $\lambda_0$  is an eigenvalue of  $A$  and the corresponding eigenspace  $\ker(\lambda_0 I - A)$  contains a positive, non-zero vector.*
- (ii) *If  $A$  is densely defined, then  $\lambda_0$  is an eigenvalue of the dual operator  $A'$  and the corresponding eigenspace  $\ker(\lambda_0 I - A')$  contains a positive, non-zero vector.*

We note in passing that in [3] the condition (3.1) is referred to as  $\mathcal{R}(\cdot, A)$  being *individually asymptotically positive at  $\lambda_0$*  if  $\lambda_0$  is a first-order pole of  $\mathcal{R}(\cdot, A)$ .

*Proof of Theorem 3.1.* (i) Let  $m \in \mathbb{N}$ ,  $m \geq 1$  denote the order of  $\lambda_0$  as a pole of  $\mathcal{R}(\cdot, A)$  and let

$$\mathcal{R}(\lambda, A) = \sum_{k=-m}^{\infty} (\lambda - \lambda_0)^k Q_k \quad (3.2)$$

be the Laurent series expansion of  $\mathcal{R}(\cdot, A)$  about  $\lambda_0$ , where  $Q_k \in \mathcal{L}(E)$ . Then  $Q_{-m} \neq 0$  and  $\text{im}(Q_{-m}) \subseteq \ker(\lambda_0 I - A)$ ; see [24, Theorem 2 in Section VIII.8]. In particular,  $\lambda_0$  is an eigenvalue of  $A$  and  $(\lambda - \lambda_0)^m \mathcal{R}(\lambda, A) \rightarrow Q_{-m}$  with respect to the operator norm as  $\lambda \downarrow \lambda_0$ . Hence Assumption (3.1) implies that  $Q_{-m}$  is a positive operator. Since  $Q_{-m}$  is non-zero its range contains a positive non-zero vector and this vector is an eigenvector of  $A$  corresponding to the eigenvalue  $\lambda_0$ .

(ii) Now assume that  $A$  is densely defined so that it has a well-defined dual operator  $A'$ . Then  $\mathcal{R}(\lambda, A') = \mathcal{R}(\lambda, A)'$  for all  $\lambda \in \rho(A) = \rho(A')$ , so it follows from (3.2) that the Laurent expansion of  $\mathcal{R}(\cdot, A')$  about  $\lambda_0$  is given by

$$\mathcal{R}(\lambda, A') = \sum_{k=-m}^{\infty} (\lambda - \lambda_0)^k Q'_k.$$

In particular, as  $Q'_{-m} \neq 0$ , the point  $\lambda_0 \in \sigma(A')$  is an  $m$ -th order pole of  $\mathcal{R}(\cdot, A')$ . As  $Q_{-m}$  is positive, so is  $Q'_{-m}$  and hence,  $\text{im}(Q'_{-m})$  contains a positive non-zero vector. As  $\text{im}(Q'_{-m}) \subseteq \ker(\lambda_0 I - A')$ , this proves the assertion as in (ii).  $\square$

Let us formulate two corollaries where Theorem 3.1 is applied to eventually positive resolvents and to eventually positive semigroups.

First we recall the definition of an eventually positive resolvent from [4, Section 8]. Let  $A: E \supseteq D(A) \rightarrow E$  be a linear operator on a complex Banach lattice  $E$  and let  $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ . We call the resolvent  $\mathcal{R}(\cdot, A)$  of  $A$  *individually eventually positive at  $\lambda_0$*  if for every  $f \in E_+$  there exists  $\lambda_1 > \lambda_0$  such that  $(\lambda_0, \lambda_1] \subseteq \rho(A)$  and  $\mathcal{R}(\lambda, A)f \geq 0$  for all  $\lambda \in (\lambda_0, \lambda_1]$ . The following corollary is an immediate consequence of Theorem 3.1.

**Corollary 3.2.** *Let  $A: E \supseteq D(A) \rightarrow E$  be a linear operator on a complex Banach lattice  $E$  and let  $\lambda_0 \in \sigma(A) \cap \mathbb{R}$  be a pole of the resolvent  $\mathcal{R}(\cdot, A)$ . Suppose that the resolvent of  $A$  is individually eventually positive at  $\lambda_0$ .*

*Then the assertions (i) and (ii) of Theorem 3.1 are fulfilled.*

To formulate the second corollary, we recall the definition of an eventually positive semigroup from [4, Section 7]. Let  $(e^{tA})_{t \geq 0}$  be a  $C_0$ -semigroup on a complex Banach lattice  $E$ . We call this  $C_0$ -semigroup *individually eventually positive* if, for every  $f \in E_+$ , there exists a time  $t_0 \geq 0$  such that  $e^{tA}f \geq 0$  for all  $t \geq t_0$ . We recall from [4, Theorem 7.6] that the spectral bound  $s(A)$  of the generator of an individually eventually positive  $C_0$ -semigroup  $(e^{tA})_{t \geq 0}$  is always contained in the spectrum unless  $s(A) = -\infty$ .

For individually eventually positive  $C_0$ -semigroups we obtain the following corollary of Theorem 3.1 which is a generalisation of [4, Theorem 7.7(a)] in that it also yields the existence of a positive eigenvector for the dual operator.



**Corollary 3.3.** *Let  $(e^{tA})_{t \geq 0}$  be an individually eventually positive  $C_0$ -semigroup on a complex Banach lattice  $E$ . Suppose that  $s(A) > -\infty$  is a pole of the resolvent  $\mathcal{R}(\cdot, A)$ .*

*Then the assertions (i) and (ii) of Theorem 3.1 are fulfilled for  $\lambda_0 = s(A)$ .*

*Proof.* Since the semigroup is individually eventually positive and since  $s(A) > -\infty$  it follows from [4, Corollary 7.3] that the resolvent of  $A$  fulfils property (3.1) in Theorem 3.1 for  $\lambda_0 = s(A)$ . Hence, the assertion follows from that theorem.  $\square$

## 4 A Perron–Frobenius theorem for resolvents

In this section we prove a Perron–Frobenius type theorem for eventually positive resolvents. In contrast to the results of Section 3 we prove not only the existence, but also the uniqueness of positive eigenvectors. Let us start by recalling that a certain Perron–Frobenius type property can be characterised by considering the spectral projection of the eigenvalue under consideration. More precisely, let  $A: E \supseteq D(A) \rightarrow E$  be a real densely defined linear operator  $A: E \supseteq D(A) \rightarrow E$  on a complex Banach lattice  $E$ ,  $\lambda_0 \in \sigma(A) \cap \mathbb{R}$  a pole of  $\mathcal{R}(\cdot, A)$  and  $u$  a quasi-interior point of  $E_+$ . A typical conclusion of such a Perron–Frobenius type theorem is:

The eigenvalue  $\lambda_0$  of  $A$  is geometrically simple and the corresponding eigenspace  $\ker(\lambda_0 I - A)$  contains a vector  $v \gg_u 0$ . Moreover, the eigenspace  $\ker(\lambda_0 I - A')$  of the dual operator contains a strictly positive functional. (4.1)

It was shown in [3, Corollary 3.3] that a very concise way of stating this conclusion is to say that the spectral projection  $P$  associated with  $\lambda_0$  fulfills  $P \gg_u 0$ . Assertion (4.1) also implies that  $\lambda_0$  is algebraically simple and the only eigenvalue with a positive eigenfunction.

It was further proved in [3, Theorem 4.4] that, under appropriate spectral assumptions combined with the domination condition (Dom),  $P \gg_u 0$  is equivalent to a certain eventual positivity property of the resolvent  $\mathcal{R}(\cdot, A)$ . On the other hand, it was demonstrated in [3, Example 5.4] that such an equivalence is no longer true if one drops the condition (Dom). However, we prove in the next theorem that some implications in [3, Theorem 4.4], namely “(ii) or (iii)  $\Rightarrow$  (i)”, remain true without (Dom).

**Theorem 4.1.** *Let  $A: E \supseteq D(A) \rightarrow E$  be a densely defined and real linear operator on a complex Banach lattice  $E$  and let  $u \in E_+$  be a quasi-interior point. Assume that  $\lambda_0 \in \sigma(A) \cap \mathbb{R}$  is a pole of the resolvent  $\mathcal{R}(\cdot, A)$  and denote the corresponding spectral projection by  $P$ . If  $\mathcal{R}(\cdot, A)$  is individually eventually strongly positive or negative with respect to  $u$  at  $\lambda_0$ , then  $P \gg_u 0$ .*

The proof of the implications “(ii) or (iii)  $\Rightarrow$  (i)” in [3, Theorem 4.4] cannot simply be adapted to work in our more general setting here. The major obstacle is that [3, Lemma 4.8] relies on the domination condition (Dom). Here, we use

a different approach which has been inspired by the proof of [1, Proposition B-III.3.5]. We also need a simple auxiliary result which was implicitly contained in the proof of [4, Lemma 7.4].

**Lemma 4.2.** *Let  $E$  be a complex Banach lattice and let  $(T_j)_{j \in J} \subseteq \mathcal{L}(E)$  an individually eventually positive net of operators, in the sense that for all  $f \in E_+$  there exists  $j_0 \in J$  such that  $T_j f \geq 0$  for all  $j \geq j_0$ . Then, for every  $f$  in the real part  $E_{\mathbb{R}}$  of  $E$ , there exists  $j_1 \in J$  such that  $|T_j f| \leq T_j |f|$  for all  $j \geq j_1$ .*

*Proof.* Choose  $j_1 \in J$  such that  $T_j f^+ \geq 0$  and  $T_j f^- \geq 0$  for all  $j \geq j_1$ . For all those  $j$  we then obtain  $T_j(|f| + f) = 2T_j f^+ \geq 0$  and  $T_j(|f| - f) = 2T_j f^- \geq 0$ . Hence,  $T_j |f| \geq -T_j f$  and  $T_j |f| \geq T_j f$  and thus  $T_j |f| \geq |T_j f|$  for all  $j \geq j_1$ .  $\square$

*Proof of Theorem 4.1.* We may assume throughout the proof that  $\lambda_0 = 0$ . Suppose that  $\mathcal{R}(\cdot, A)$  is individually eventually strongly positive with respect to  $u$  at  $\lambda_0 = 0$ . We are going to show that (4.1) is fulfilled.

According to Corollary 3.2 we can find vectors  $0 < v \in \ker A$  and  $0 < \varphi \in \ker A'$ . We observe that every element  $0 < w \in \ker A$  fulfils  $w \gg_u 0$ . Indeed, by assumption, for each such  $w$  we can find a number  $\lambda > 0$  for which  $\lambda \in \rho(A)$  and  $w = \lambda \mathcal{R}(\lambda, A)w \gg_u 0$ . Therefore,  $v \gg_u 0$ .

Next we show that the functional  $\varphi$  is strictly positive. For every  $0 < f \in E$  we can find a number  $0 < \lambda \in \rho(A)$  such that  $\lambda \mathcal{R}(\lambda, A)f \gg_u 0$ ; in particular,  $\lambda \mathcal{R}(\lambda, A)f$  is a quasi-interior point of  $E_+$ . Hence,

$$\langle \varphi, f \rangle = \langle \lambda \mathcal{R}(\lambda, A')\varphi, f \rangle = \langle \varphi, \lambda \mathcal{R}(\lambda, A)f \rangle > 0.$$

Thus,  $\varphi$  is indeed strictly positive.

It remains to show that  $\ker A$  is one-dimensional. To this end, we first prove that  $E_{\mathbb{R}} \cap \ker A$  is a sublattice of the real part  $E_{\mathbb{R}}$  of  $E$ . Fix  $w \in E_{\mathbb{R}} \cap \ker A$ . According to Lemma 4.2 we can find a number  $0 < \lambda \in \rho(A)$  such that  $|w| = \lambda |\mathcal{R}(\lambda, A)w| \leq \lambda \mathcal{R}(\lambda, A)|w|$ . By testing the positive vector  $\lambda \mathcal{R}(\lambda, A)|w| - |w|$  against the strictly positive functional  $\varphi \in \ker A'$  we obtain

$$\langle \varphi, \lambda \mathcal{R}(\lambda, A)|w| - |w| \rangle = \langle \lambda \mathcal{R}(\lambda, A')\varphi, |w| \rangle - \langle \varphi, |w| \rangle = 0$$

and thus,  $\lambda \mathcal{R}(\lambda, A)|w| = |w|$ . This proves that  $|w| \in E_{\mathbb{R}} \cap \ker A$ , so  $E_{\mathbb{R}} \cap \ker A$  is indeed a sublattice of  $E_{\mathbb{R}}$ .

We have seen above that every non-zero positive vector in  $w \in \ker A$  fulfils  $w \gg_u 0$  and is thus a quasi-interior point of  $E_{\mathbb{R}}$ . Hence, according to [21, Corollary 2 to Theorem II.6.3],  $v$  is also a quasi-interior point of the positive cone of the Banach lattice  $E_{\mathbb{R}} \cap \ker A$  (when endowed with the norm inherited from  $E_{\mathbb{R}}$ ). We have thus shown that every positive non-zero element of the real Banach lattice  $E_{\mathbb{R}} \cap \ker A$  is a quasi-interior point of its positive cone. This implies that  $E_{\mathbb{R}} \cap \ker A$  is one-dimensional; see [17, Lemma 5.1] or [11, Remark 5.9]. Since  $A$  is real, we have  $\ker A = E_{\mathbb{R}} \cap \ker A + iE_{\mathbb{R}} \cap \ker A$ , so we conclude that  $\ker A$  is one-dimensional over the complex field. This proves (4.1).

Now assume instead that  $\mathcal{R}(\cdot, A)$  is individually eventually strongly negative with respect to  $u$  at  $\lambda_0 = 0$ . Then the resolvent of  $-A$ , which is given by

$\mathcal{R}(\lambda, -A) = -\mathcal{R}(-\lambda, A)$  for all  $\lambda \in \rho(-A) = -\rho(A)$ , is individually eventually strongly positive with respect to  $u$  at 0. Hence, by what we have just seen, the spectral projection of  $-A$  associated with 0 is strongly positive with respect to  $u$ . This spectral projection coincides with  $P$ , which proves the assertion by what we have shown above.  $\square$

## 5 A Perron–Frobenius theorem for semigroups

In this final section we pursue a similar goal as in Section 4, but this time for eventually positive semigroups instead of resolvents. In [3, Theorem 5.2] it was shown that, under some assumptions which include the smoothing condition (Smo), individual eventual strong positivity with respect to  $u$  of a semigroup  $(e^{tA})_{t \geq 0}$  is equivalent to a certain spectral condition that includes the Perron–Frobenius properties discussed at the start of the previous section. As demonstrated in [3, Example 5.4] this results fails in general if the smoothing condition (Smo) is dropped. However, we are now going to prove that at least a certain part of [3, Theorem 5.2] remains true without the condition (Smo).

**Theorem 5.1.** *Let  $(e^{tA})_{t \geq 0}$  be a real  $C_0$ -semigroup on a complex Banach lattice  $E$  and let  $u \in E_+$  be a quasi-interior point. Assume that  $s(A)$  is not equal to  $-\infty$  and a pole of the resolvent  $\mathcal{R}(\cdot, A)$ . If  $(e^{tA})_{t \geq 0}$  is individually eventually strongly positive with respect to  $u$ , then the spectral projection  $P$  corresponding to  $s(A)$  fulfils  $P \gg_u 0$ .*

For a similar reason as in Section 4 we cannot simply modify the relevant part of the proof of [3, Theorem 5.2]. Instead we adapt the argument in the proof of Theorem 4.1 for semigroups.

*Proof of Theorem 5.1.* We may assume throughout that  $s(A) = 0$ . According to Corollary 3.3 there exists a vector  $0 < v \in \ker A$  and a functional  $0 < \varphi \in \ker A'$ . To prove (4.1) we now proceed similarly as in the proof of Theorem 4.1 with  $\lambda_0 = s(A) = 0$ .

First note that every vector  $0 < w \in \ker A$  fulfils  $w \gg_u 0$ . Indeed, for each such vector we can find a time  $t \geq 0$  for which we have  $w = e^{tA}w \gg_u 0$ . In particular we have  $v \gg_u 0$ . Next we prove that the functional  $\varphi$  is strictly positive. To this end, let  $0 < x \in E$ . We can find a time  $t \geq 0$  such that  $e^{tA}x \gg_u 0$ , so  $e^{tA}x$  is a quasi-interior point of  $E_+$ . Hence, as  $\varphi$  is non-zero we obtain

$$\langle \varphi, x \rangle = \langle (e^{tA})' \varphi, x \rangle = \langle \varphi, e^{tA}x \rangle > 0,$$

which shows that  $\varphi$  is indeed strictly positive.

To conclude the proof, we still have to show that  $\ker A$  is one-dimensional. As in the proof of Theorem 4.1, let us first show that  $E_{\mathbb{R}} \cap \ker A$  is a sublattice of  $E_{\mathbb{R}}$ . So, take  $w \in E_{\mathbb{R}} \cap \ker A$  and choose a time  $t_1 \geq 0$  such that  $|w| = |e^{tA}w| \leq e^{tA}|w|$  for all  $t \geq t_1$ ; such a time  $t_1$  exists according to Lemma 4.2. For  $t \geq t_1$  we test

the positive vector  $e^{tA}|w| - |w|$  against the strictly positive functional  $\varphi \in \ker A'$ , thus obtaining

$$\langle \varphi, e^{tA}|w| - |w| \rangle = \langle (e^{tA})'\varphi, |w| \rangle - \langle \varphi, |w| \rangle = 0$$

and hence  $e^{tA}|w| = |w|$ . For every  $t \geq 0$  this implies  $e^{tA}|w| = e^{tA}e^{t_1A}|w| = e^{(t+t_1)A}|w| = |w|$ . Therefore,  $|w| \in \ker A$ , so  $E_{\mathbb{R}} \cap \ker A$  is indeed a sublattice of  $E_{\mathbb{R}}$ . Now the same arguments as in the proof of Theorem 4.1 show that  $\ker A$  is indeed one-dimensional.  $\square$

If the peripheral spectrum of  $A$  is finite and consists of poles of the resolvent and if the smoothing condition (Smo) is fulfilled, then [3, Theorem 5.2] asserts, among other things, that individual eventual strong positivity of  $(e^{tA})_{t \geq 0}$  with respect to  $u$  implies that the semigroup  $(e^{t(A-s(A))})_{t \geq 0}$  is bounded. It is an interesting question whether this result remains true without the condition (Smo). This does not even seem to be clear if the semigroup is strongly positive with respect to  $u$ , that is, if  $e^{tA} \gg_u 0$  for all  $t > 0$ .

If, however, the semigroup under consideration is eventually norm continuous, then the situation is much simpler. In this case we obtain the following corollary which shows that the implication “(i)  $\Rightarrow$  (ii)” in [3, Corollary 5.3] is true under weaker assumptions than stated there.

**Corollary 5.2.** *Let  $(e^{tA})_{t \geq 0}$  be a real and eventually norm-continuous  $C_0$ -semigroup on a complex Banach lattice  $E$  and let  $u \in E_+$  be a quasi-interior point. Assume that  $s(A) > -\infty$  and that the peripheral spectrum of  $A$  is finite and consists of poles of the resolvent.*

*If  $(e^{tA})_{t \geq 0}$  is individually eventually strongly positive with respect to  $u$ , then the rescaled semigroup  $(e^{t(A-s(A))})_{t \geq 0}$  is bounded, the spectral bound  $s(A)$  is a dominant spectral value of  $A$  and the corresponding spectral projection  $P$  fulfils  $P \gg_u 0$ .*

*Proof.* We may assume that  $s(A) = 0$ . First recall from [4, Theorem 7.6] that  $s(A)$  is a spectral value of  $A$ . It follows from Theorem 5.1 that  $P \gg_u 0$ . Hence,  $s(A)$  is a first order pole of  $\mathcal{R}(\cdot, A)$  according to [3, Corollary 3.3], and this in turn implies that  $\sigma_{\text{per}}(A)$  consists of first order poles of the resolvent; see [4, Theorem 7.7(ii)].

Now, let  $\sigma_{\text{per}}(A) = \{i\beta_1, \dots, i\beta_n\}$  and denote by  $Q \in \mathcal{L}(E)$  the spectral projection of  $A$  associated with  $\sigma_{\text{per}}(A)$ . Since  $\sigma_{\text{per}}(A)$  consists of first order poles of the resolvent, we have  $QE = \bigoplus_{k=1}^n \ker(i\beta_k I - A)$ . Hence, the semigroup  $(e^{tA})_{t \geq 0}$  is bounded on the range of  $Q$ . On the other hand, since the semigroup is eventually norm continuous and since  $\sigma_{\text{per}}(A)$  is isolated from the rest of spectrum by assumption, it follows from [7, Theorem II.4.18] that the spectral bound of  $A|_{\ker Q}$  fulfils  $s(A|_{\ker Q}) < 0$ . Using again that our semigroup is eventually norm continuous, we conclude from [7, Corollary IV.3.11] that  $e^{tA} \rightarrow 0$  on  $\ker Q$  with respect to the operator norm as  $t \rightarrow \infty$ . Hence,  $(e^{tA})_{t \geq 0}$  is indeed bounded as claimed.

Finally, the boundedness and the individual eventual positivity of  $(e^{tA})_{t \geq 0}$  imply immediately that this semigroup is *individually asymptotically positive*, see [3, Definition 8.1]. Hence, it follows from [3, Theorem 8.3] that  $s(A)$  is a dominant spectral value of  $A$ .  $\square$

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